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Novel Estimates of Hermite-Hadamard Type Inequalities and Midpoint-like Results for Quantum Integrals

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Abstract

In this study, using extended convex functions, we present and rigorously derive several new estimates of Hermite-Hadamard type inequalities in the context of quantum integrals. We apply power mean inequality and Hölder's inequality to extend these bounds further. As consequence of these results, we attain $q-$midpoint-like, Simpson's like, averaged midpoint-like, and trapezoid type results. In order to verify our presented results, we express those with the help of graphs by certain choices of parameters involved. In the end, to focus on utilization of the results, we offer two novel applications namely averaged midpoint-trapezoid-like and $q-$trapezoid like results of the produced results.

Keywords: Hadamard inequalities; midpoint-like results; quantum integrals.

1 Introduction and Historical Background

Convexity is one of the most fundamental and widely studied concepts in mathematical analysis, optimization, and related fields. With origins in classical results in convexity and integral calculus, the study of inequalities has long been a prominent and active field in mathematical analysis. A key tool that offers sophisticated and potent constraints for the integral average of convex functions over a finite interval is the Hermite-Hadamard inequality [19, 18]. This classical inequality was investigated by [25, 24] for its bound and further refinement were offered by [28, 27]. In order to extend the bounds to Ostrowski type inequalities for $n-$times differentiable function Cerone et al. [13] investigated the Hermite-Hadamard inequality. In addition to providing a theoretical basis for convex analytic advances, this inequality is crucial for numerical approximation, optimization, and other applied domains.

A key idea in analysis and optimization, convexity provides a geometric concept that is easy to understand and allows for a variety of potent outcomes. The property that the value of function at any point on the line segment connecting two locations is less than or equal to the corresponding convex combination of the function values at those places is satisfied by classical convex functions. There are several new directions in the context of convex functions like $m-$convex [39], $h-$convex [40], $s-$convex [20] and $(s, m)-$convex [5] functions for investigation of different inequalities. In particular Dragomir [14] and Dragomir and Toader [16] analyzed the bounds of H-H inequality for $m-$convex functions and further investigated in [15, 17] for the well known $s-$convexity. Motivated by [35], Nawaz et al. [32] examined inequality (1) to calculate its novel bounds for product type convex functions. The fractional version of H-H inequality and its bounds for stochastic process preserving convexity are discussed in [26, 31]. By removing the limit restrictions from conventional calculus and discretizing the path taken by the integrals, the calculus without limits known as $q-$calculus has had a significant impact on modern mathematics, physics, and mechanics.

The $q-$integral is not a quantum mechanical idea, but rather a mathematical tool that has found widespread application since it naturally emerges in and provides solutions to issues in a variety of domains. The $q-$integral offers a discrete, summation-based substitute for the Riemann integral in contemporary analysis, providing a distinctive viewpoint for investigating asymptotic analysis, understanding function spaces, and establishing inequalities. Bounding the value of an integral, sometimes by linking it to the values of the function at particular places (such as ends or the midpoint), is the main goal of the field of integral inequalities. The Midpoint Trapezoidal Rule error bound, or the classical Midpoint Inequality, for a twice-differentiable function employing $q-$analogues. Since they give error boundaries for numerical integration techniques in the context of $q-$calculus, these $q-$midpoint inequalities are essential, particularly for functions that naturally occur in $q-$series and $q-$combinatorics. Although Euler's fundamental idea on $q-$calculus gave rise to this theory late in the 18th century, it only recently began to grow geometrically in various directions, including $q-$mechanics, combinatorics, number theory, hyper geometric Series, and cryptography.

The $q-$derivative and $q-$integral concepts were created by Jackson [23], who also looked at the outcomes of the $q-$difference equations. In the context of $q-$calculus, several academics looked into Hermite-Hadamard, Simpson, Trapezoid, Hölder, and Gruss type inequalities. For the Fejer type bounds of H-H inequality, Barsam et al. [6] utilized the uniformly convex functions to analyze quantum integrals. Butt et al. [9, 11] used symmetric quantum integral and quantum integrals respectively to establish bounds on finite rectangular plane and cyclic Jensen type bounds for weighted H-H inequality and Montgomery identities. Further investigation on cyclic Jensen type inequalities can be witnessed in [36]. Recently, Tariboon and Ntouyas [37, 38] explored the

$q-$derivative and $q-$integral by utilizing a completely new methodology and re-framing the traditional integral inequalities.

By focusing on initial value problems, the authors were able to replicate existence and uniqueness results for first order and second order impulsive difference equations. Furthermore, Zhao et al. [46] examined applications of generalized quantum integrals for differentiable convex functions. They analyzed quantum integrals involving parameter by utilizing left and right quantum derivatives to establish new estimates for ostrowski's, Simpson's, $q-$midpoint-like and trapezoid type integrals. The well known two folded inequality known as Hermite-Hadamard inequality for convex functions over $\mathbb{R}$ may be given as,

$$P\left(\frac{\xi+\eta}{2}\right) \leq \frac{1}{\eta-\xi}\int_{\xi}^{\eta} P(x)dx \leq \frac{P(\xi)+P(\eta)}{2}. \tag{1}$$

Inequality (1) has been extensively probed for its new estimates for convex functions, generalized convex functions, invex, pre-invex functions, fractional integrals and quantum integrals using a variety of operators.

The authors [41, 44] established new bounds for inequality (1) for the functions whose derivative is $(\delta, m)-$convex and further studied in [4, 7] for refinement of bounds. Noor et al. [33] further studied convex functions by introducing $(\delta, m, h)-$convex functions and established novel estimates for inequality (1). Zhang at el. [45], using the $(\delta, m)-$convex functions, calculated new estimates for quantum integrals using a novel approach. As a special case of classical inequality (1), Butt et al. [10, 12] explored the new variants of quantum mid-point type inequalities and Jensen Mercer type mid-point inequalities. Motivated by [30, 8] and [22, 42] along with [21], with the help of generalized convex functions, we examine inequality (1) for its novel bounds by taking into account the quantum integrals. Further, we extend these results by power mean inequality and Hölder's inequality. We estimate the bounds of different midpoint-like results and express the results graphically to justify the validity. To conclude, we offer two new applications of the presented results.

The structure of the paper is as follows: Section 1 contains an introduction and the foundational literature that is important for the current study. New estimates for various types of quantum integrals followed by graphs are the focus of Section 2. In Section 3, we construct $q-$midpoint-like, $q-$Simpson's-like, $q-$averaged-midpoint-like, and $q-$trapezoid-like midpoint results as outcomes of the results that have been presented. In the end, a brief conclusion along with new areas of research connected to our work are proposed.

2 Preliminaries

In this section is some well known definitions and results from literature are reproduced. Toader [39] introduced the definition of $m-$convex function.

Definition 2.1. *A function $P : I \to (0, \infty)$ is $m-$convex, if,*

$$P(\eta r_1 + m(1-\eta)r_2) \leq \eta P(r_1) + m(1-\eta)P(r_2), \tag{2}$$

for all $r_1, r_2 \in I$, $m \in (0, 1]$, $\eta \in [0, 1]$.

Mihesan [29] generalized $m-$convex functions by $(\delta, m)-$convex function as follows;

Definition 2.2. *A function $P : I \to (0, \infty)$ is $(\delta, m)-$convex, if,*

$$P(\eta r_1 + m(1-\eta)r_2) \leq \eta^\delta P(r_1) + m(1-\eta^\delta)P(r_2), \tag{3}$$

for all $r_1, r_2 \in I$, $\eta \in [0, 1]$, $(\delta, m) \in [0, 1]^2$.

Ozdemir et al. [34] introduced the idea of $(h, m)-$convex function as follows;

Definition 2.3. *A function $P : I \to (0, \infty)$ is $(h, m)-$convex, if,*

$$P(\eta r_1 + m(1-\eta)r_2) \leq h(\eta)P(r_1) + mh(1-\eta)P(r_2), \tag{4}$$

for all $r_1, r_2 \in I$, $\eta \in [0, 1]$, and $h : [0, 1] \to [0, 1]$.

Noor et al. [33] re-framed idea of $m-$convexity and $(\delta, m)-$convexity with the definition of $(\delta, m, h)-$convexity.

Definition 2.4. *A function $P : I \to (0, \infty)$ is $(\delta, m, h)-$convex, if,*

$$P(\eta r_1 + m(1-\eta)r_2) \leq h(\eta^\delta)P(r_1) + mh(1-\eta^\delta)P(r_2),$$

for all $r_1, r_2 \in I, \eta \in [0, 1], (\delta, m) \in (0, 1]^2$ and $h : (0, 1] \to (0, 1]$.

Definition 2.5. *The well reputed $q-$Jackson integral [23] on an interval $[\xi_1, \xi_2]$ is defined as,*

$$\int_{\xi_1}^{\xi_2} P(\eta)d_q\eta = \int_0^{\xi_2} P(\eta)d_q\eta - \int_0^{\xi_1} P(\eta)d_q\eta.$$

Definition 2.6. [37] *The $q-$derivative of a function is characterized as,*

$${}_rD_qP(\eta) = \frac{P(\eta) - P(q\eta + (1-q)r)}{(1-q)(\eta - r)}, \eta \neq r.$$

Definition 2.7. [37] *For continuous function $P : I \to \mathbb{R}$, $q-$integral is given as,*

$$\int_r^\xi P(\eta)_r d_q\eta = (1-q)(\xi - r)\sum_{k=0}^{\infty} q^k P(q^k\xi + (1-q^k)r).$$

Moreover, if $a \in (r, \xi)$ then,

$$\int_a^\xi P(\eta)_r d_q\eta = \int_r^\xi P(\eta)_r d_q\eta - \int_r^a P(\eta)_r d_q\eta.$$

Following lemma plays a substantial role for our results.

Lemma 2.1. [45] *Let $P : I \to \mathbb{R}$ be a continuous and $q-$differentiable mapping on I^o with $q \in (0, 1)$. Further assume that ${}_rD_q$ is integer-able on I, then for all $t_1, t_2 \in [0, 1]$, the following identity holds,*

$$\Omega \leq (\xi - r)\Bigg\{\int_0^{t_2} (q\eta + t_1t_2 - t_1)_r D_qP\big(\eta\xi + (1-\eta)r\big)_0 d_q\eta$$
$$+ \int_{t_2}^1 (q\xi + t_1t_2 - 1)_r D_qP(\xi\eta + (1-\xi)r)_0 d_q\eta\Bigg\},$$

where

$$\Omega = t_1\big[t_2P(\xi) + (1-t_2)P(r)\big] + (1-t_1)P\big(t_2\xi + (1-t_2)r\big) - \frac{1}{\xi - r}\int\limits_r^{\xi} P(x)_r d_q x.$$

3 Main Results

We are in position now to discuss the main results of this study. This section comprises the results of our work regarding new estimates of several integrals involving $q-$derivatives and $q-$integrals.

Theorem 3.1. *Let $P : \left[r, \dfrac{\xi}{m}\right] \to \mathbb{R}$ be a continuous and $q-$differentiable function on $\left[r, \dfrac{\xi}{m}\right]$ along with $(\delta, m, h)-$convexity. If $|_rD_qP|$ is integer-able on $\left[r, \dfrac{\xi}{m}\right]$ with $0 < q < 1$, then the following inequality holds for all $t_1, t_2 \in [0,1]$ and $m \in (0,1]$,*

$$\begin{aligned}|\Omega| \leq (\xi - r)\Bigg\{&\Bigg[\int\limits_0^{t_2} h\big(\eta^{\delta}\big)\,|q\eta - (t_1 - t_1t_2)|\ {}_0d_q\eta + \int\limits_0^{1} h\big(\eta^{\delta}\big)\,|q\eta - (1 - t_1t_2)|\ {}_0d_q\eta \\ &- \int\limits_0^{t_2} h\big(\eta^{\delta}\big)\,|q\eta - (1 - t_1t_2)|\ {}_0d_q\eta\Bigg]\,|\,{}_rD_qP(\xi)| \\ &+ m\Bigg[\int\limits_0^{t_2} h\big(1-\eta^{\delta}\big)\,|q\eta - (t_1 - t_1t_2)|\ {}_0d_q\eta + \int\limits_0^{1} h\big(1-\eta^{\delta}\big)\,|q\eta - (1 - t_1t_2)|\ {}_0d_q\eta \\ &- \int\limits_0^{t_2} h\big(1-\eta^{\delta}\big)|q\eta - (1 - t_1t_2)|_0d_q\eta\Bigg]\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\Bigg\}. \end{aligned} \tag{5}$$

Proof. Application of modulus property in Lemma 2.1, we have

$$\begin{aligned}|\Omega| \leq (\xi - r)\Bigg\{&\int\limits_0^{t_2} |q\eta + t_1t_2 - t_1|\,\big|{}_rD_qP\big(\eta\xi + (1-\eta)r\big)\big|_0\, d_q\eta \\ &+ \int\limits_{t_2}^{1} |q\xi + t_1t_2 - 1|\,|{}_rD_qP(\xi\eta + (1-\xi)r)|\ {}_0d_q\eta\Bigg\}.\end{aligned}$$

Since $|_rD_qP|$ is $(\delta, m, h)-$convex, so we have

$$\begin{aligned}|\Omega| \leq (\xi - r)\Bigg\{&\int\limits_0^{t_2} |q\eta - (t_1 - t_1t_2)|\,\Big[h\big(\eta^{\delta}\big)\big|{}_rD_qP(\xi)\big| + mh\big(1-\eta^{\delta}\big)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\ {}_0d_q\eta\Big] \\ &+ \int\limits_{t_2}^{1} |q\eta - (1 - t_1t_2)|\,\Big[h\big(\eta^{\delta}\big)\,|{}_rD_qP(\xi)| + mh\big(1-\eta^{\delta}\big)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\Big]\ {}_0d_q\eta\Bigg\}\end{aligned}$$

$$= (\xi - r)\left\{\int_0^{t_2} |q\eta - (t_1 - t_1t_2)| \left[h\big(\eta^\delta\big)\left|{}_rD_qP(\xi)\right| + mh\big(1-\eta^\delta\big)\left|{}_rD_qP\left(\frac{r}{m}\right)\right| \ {}_0d_q\eta\right]\right.$$
$$+ \int_0^1 |q\eta - (1 - t_1t_2)| \left[h\big(\eta^\delta\big)\left|{}_rD_qP(\xi)\right| + mh\big(1-\eta^\delta\big)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\right] {}_0d_q\eta$$
$$\left.- \int_0^{t_2} |q\eta - (1 - t_1t_2)|\left[h\big(\eta^\delta\big)\left|{}_rD_qP(\xi)\right| + mh\big(1-\eta^\delta\big)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\right] {}_0d_q\eta\right\}.$$

Finally, we obtain,

$$|\Omega| \leq (\xi - r)\left\{\left[\int_0^{t_2} h\big(\eta^\delta\big)|q\eta - (t_1 - t_1t_2)| \ {}_0d_q\eta + \int_0^1 h\big(\eta^\delta\big)|q\eta - (1 - t_1t_2)| \ {}_0d_q\eta\right.\right.$$
$$\left.- \int_0^{t_2} h\big(\eta^\delta\big)\big|q\eta - (1 - t_1t_2)\big| \ {}_0d_q\eta\right] |\ {}_rD_qP(\xi)| + m\left[\int_0^{t_2} h\big(1-\eta^\delta\big)|q\eta - (t_1 - t_1t_2)| \ {}_0d_q\eta\right.$$
$$\left.\left. + \int_0^1 h\big(1-\eta^\delta\big)|q\eta - (1 - t_1t_2)| \ {}_0d_q\eta - \int_0^{t_2} h\big(1-\eta^\delta\big)|q\eta - (1 - t_1t_2)|_0d_q\xi\right]\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\right\}.$$

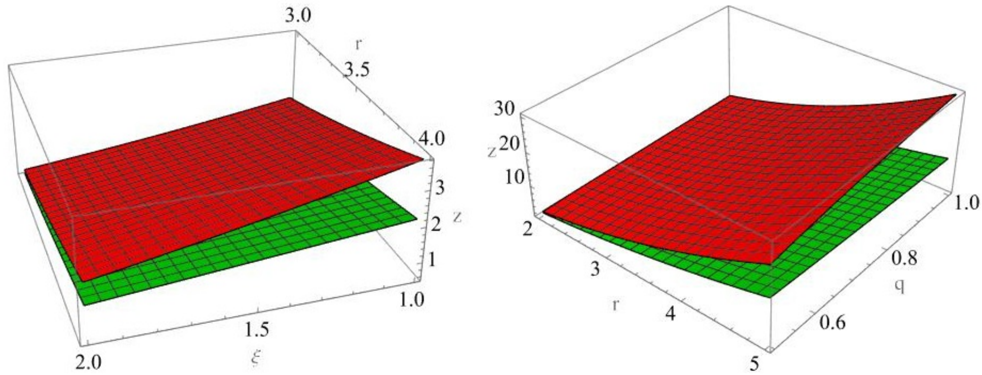
□

Corollary 3.1. *For $h(t) = t^s$ in (5), we obtain*

$$|\Omega| \leq (\xi - r)\left\{\left[\int_0^{t_2} \eta^{s\delta}|q\eta - (t_1 - t_1t_2)| \ {}_0d_q\eta + \int_0^1 \big(\eta^\delta\big)^s|q\eta - (1 - t_1t_2)| \ {}_0d_q\eta\right.\right.$$
$$\left.- \int_0^{t_2} \eta^{s\delta}\big|q\eta - (1 - t_1t_2)\big| \ {}_0d_q\eta\right] \big|\ {}_rD_qP(\xi)\big| + m\left[\int_0^{t_2} \big(1-\eta^\delta\big)^s|q\eta - (t_1 - t_1t_2)| \ {}_0d_q\eta\right.$$
$$\left.\left. + \int_0^1 \big(1-\eta^\delta\big)^s|q\eta - (1 - t_1t_2)| \ {}_0d_q\eta - \int_0^{t_2} \big(1-\eta^\delta\big)^s|q\eta - (1 - t_1t_2)|_0d_q\eta\right]\left|{}_rD_qP\left(\frac{r}{m}\right)\right|\right\}, \quad (6)$$

which are the estimates of quantum integral for $(\delta, s, m)-$convex function.

Example 3.1. *Consider $P(x) = x^2$ and $h \equiv I$, then P is $(\delta, m, h)-$convex. By making use of $(\delta, m, h)-$convexity, we calculate all the integrals involved in (5). We draw few graphs for various values of parameters involved and present in Figure 1. Left hand side of (5) is represented by green color and right side by red color. Clearly, (5) is justified by graphs.*



(A) Here $t_1 = 0.5, t_2 = 0.1, q = 0.7, \xi \in [1, 2]$ and $r \in [3, 4]$

(B) Here $t_1 = 0.5, t_2 = 0.1, \xi = 1, r \in [2, 5]$ and $q \in [0.5, 1]$

Figure 1: Graphical illustration of (5).

Remark 3.1.

1. *For $\delta = 1$ in (6), we obtain the bounds for $(s, m)-$convex function.*
2. *For $\delta = 1$ in (5), we recapture Theorem 4 in [1].*
3. *Taking h to be identity function in (5), we obtain Theorem 3.2 in [45].*

Theorem 3.2. *Let $P : \left[r, \dfrac{\xi}{m}\right] \to \mathbb{R}$ be a continuous and $q-$differentiable function on $\left[r, \dfrac{\xi}{m}\right]$. If ${}_rD_qP$ is integer-able on $\left[r, \dfrac{\xi}{m}\right]$ with $0 < q < 1$, then the following inequality holds for all $t_1, t_2 \in [0, 1]$ and $|\,{}_rD_qP|^\beta$ to be $(\delta, m, h)-$convex for $\beta \geq 1$ along with $m \in (0, 1]$:*

$$|\Omega| \leq (\xi - r)\left\{\left[\int_0^1 |q\eta - (1 - t_1t_2)|\,{}_0d_q\eta\right]^{1-\frac{1}{\beta}} \times \left[\left(\int_0^1 h(\eta^\delta)\,|q\eta - (1 - t_1t_2)|\,{}_0d_q\eta\right)\left|\,{}_rD_qP(\xi)\right|^\beta\right.\right.$$
$$\left.+m\left(\int_0^1 h(1 - \eta^\delta)\,|q\eta - (1 - t_1t_2)|_0 d_q\eta\right)\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]^{\frac{1}{\beta}} + (1 - t_1)t_2^{1-\frac{1}{\beta}}$$
$$\left.\times\left[\left(\int_0^{t_2} h(\eta^\delta)\,{}_0d_q\eta\right)\left|\,{}_rD_qP(\xi)\right|^\beta + m\left(\int_0^{t_2} h(1 - \eta^\delta)\,{}_0d_q\eta\right)\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]^{\frac{1}{\beta}}\right\}. \tag{7}$$

Proof. Using Power Mean inequality along with modulus property in Lemma 2.1,

$$|\Omega| \leq (\xi - r)\left\{\left(\int_0^1 |q\eta - (1 - t_1t_2)|\,{}_0d_q\eta\right)^{1-\frac{1}{\beta}}\left(\int_0^1 |q\eta - (1 - t_1t_2)|\left|\,{}_rD_qP\big(\eta\xi + (1 - \eta)r\big)\right|^\beta\,{}_0d_q\eta\right)^{\frac{1}{\beta}}\right.$$
$$\left.+(1 - t_1)\left(\int_0^{t_2} 1\,{}_0d_q\eta\right)^{1-\frac{1}{\beta}}\left(\int_0^{t_2}\left|\,{}_rD_qP\left(\eta\xi + (1 - \eta)\,r\right)\right|^\beta\,{}_0d_q\eta\right)^{\frac{1}{\beta}}\right\}. \tag{8}$$

Now, utilizing the fact that $\left|{}_{r}D_qP\right|^{\beta}$ is $(\delta,m,h)-$convex, we have

$$\begin{aligned}
&\int_0^1 |q\eta-(1-t_1t_2)|\left|\,{}_rD_qP\big(\eta\xi+(1-\eta)r\big)\right|^{\beta}\,{}_0d_q\eta\\
&\leq \int_0^1 |q\eta-(1-t_1t_2)|\left[h(\eta^{\delta})\,|\,{}_rD_qP(\xi)|^{\beta}+mh(1-\eta^{\delta})\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\,{}_0d_q\eta\right]\\
&=\left[\int_0^1 h(\eta^{\delta})|q\eta-(1-t_1t_2)|\,{}_0d_q\eta\right]\left|\,{}_rD_qP(\xi)\right|^{\beta}\\
&\qquad+m\left[\int_0^1 h(1-\eta^{\delta})|q\eta-(1-t_1t_2)|\,{}_0d_q\eta\right]\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}. \qquad (9)
\end{aligned}$$

Similarly,

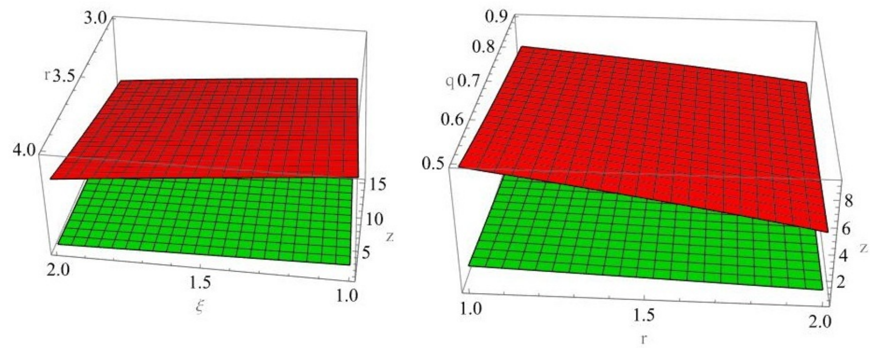
$$\begin{aligned}
&\int_0^{t_2}\left|\,{}_rD_qP\big(\eta\xi+(1-\eta)r\big)\right|^{\beta}\,{}_0d_q\eta\\
&\leq \int_0^{t_2}\left[h(\eta^{\delta})\,|{}_rD_qP\,(\xi)|^{\beta}+mh(1-\eta^{\delta})\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]\,{}_0d_q\eta\\
&=|{}_rD_qP\,(\xi)|^{\beta}\int_0^{t_2}h(\eta^{\delta})\,{}_0d_q\eta+m\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\int_0^{t_2}h(1-\eta^{\delta})\,{}_0d_q\eta. \qquad (10)
\end{aligned}$$

By making use of (9) and (10) in (8), we obtain,

$$\begin{aligned}
|\Omega|\leq(\xi-r)&\left\{\left(\int_0^1|q\eta-(1-t_1t_2)|\,{}_0d_q\eta\right)^{1-\frac{1}{\beta}}\left[\left(\int_0^1 h(\eta^{\delta})|q\eta-(1-t_1t_2)|\,{}_0d_q\eta\right)\left|\,{}_rD_qP(\xi)\right|^{\beta}\right.\right.\\
&\left.+m\left(\int_0^1 h(1-\eta^{\delta})|q\eta-(1-t_1t_2)|_0d_q\eta\right)\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]^{\frac{1}{\beta}}\\
&\left.+(1-t_1)t_2^{1-\frac{1}{\beta}}\left[\left(\int_0^{t_2}h(\eta^{\delta})\,{}_0d_q\eta\right)\left|\,{}_rD_qP(\xi)\right|^{\beta}+m\left(\int_0^{t_2}h(1-\eta^{\delta})\,{}_0d_q\eta\right)\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]^{\frac{1}{\beta}}\right\}.
\end{aligned}$$

□

Example 3.2. *Under the assumptions of Example 3.1, by making use of $(\delta,m,h)-$convexity, we calculate all the integrals involved in (7). We draw few graphs for various values of parameters involved and present in Figure 2. Left hand side of (7) is represented by green color and right side by red color. Clearly, (7) is justified by graphs.*



(A) Here $t_1 = 0.1, t_2 = 0.5, q = 0.7, \xi \in [1, 2]$ and $r \in [3, 4]$ (B) Here $t_1 = 0.1, t_2 = 0.5, \xi = 3, r \in [1, 2]$ and $q \in [0.5, 0.9]$

Figure 2: Graphical illustration of (7).

Corollary 3.2. *For $h(t) = t^s$ in Theorem 3.2, we have*

$$|\Omega| \leq (\xi - r) \left\{ \left[\int_0^1 |q\eta - (1 - t_1t_2)| \, {}_0d_q\eta \right]^{1-\frac{1}{\beta}} \left[\left(\int_0^1 \left(\eta^{\delta}\right)^s |q\eta - (1 - t_1t_2)| \, {}_0d_q\eta \right) \left| {}_rD_qP(\xi) \right|^{\beta} \right.\right.$$
$$\left. + m \left(\int_0^1 \left(1 - \eta^{\delta}\right)^s |q\eta - (1 - t_1t_2)|_0 d_q\eta \right) \left| {}_rD_qP\left(\frac{r}{m}\right) \right|^{\beta} \right]^{\frac{1}{\beta}} + (1 - t_1)t_2^{1-\frac{1}{\beta}}$$
$$\left. \times \left[\left(\int_0^{t_2} \left(\eta^{\delta}\right)^s \, {}_0d_q\eta \right) \left| {}_rD_qP(\xi) \right|^{\beta} + m \left(\int_0^{t_2} \left(1 - \eta^{\delta}\right)^s \, {}_0d_q\eta \right) \left| {}_rD_qP\left(\frac{r}{m}\right) \right|^{\beta} \right]^{\frac{1}{\beta}} \right\}. \tag{11}$$

Inequality (11) represents the estimates of $(\delta, s, m)-$convex functions.

Remark 3.2. *By setting $\delta = 1$ in (11), we obtain bounds for $(s, m)-$convex functions.*

Theorem 3.3. *Let $P : \left[r, \frac{\xi}{m}\right] \to \mathbb{R}$ be a continuous and $q-$differentiable function on $\left[r, \frac{\xi}{m}\right]$ along with the assumption that $\left|{}_rD_qP\right|^{\beta}$, $\beta > 1$ is $(\delta, m, h)-$convex. Then, for $m \in (0, 1]$, the following inequality holds,*

$$|\Omega| \leq (\xi - r) \left\{ \left(\int_0^1 |q\eta - (1 - t_1t_2)|^{\mu} \, {}_0d_q\eta \right)^{\frac{1}{\mu}} \right.$$
$$\times \left(\int_0^1 \left[h\left(\eta^{\delta}\right) \left| {}_rD_qP(\xi) \right|^{\beta} + mh\left(1 - \eta^{\delta}\right) \left| {}_rD_qP\left(\frac{r}{m}\right) \right|^{\beta} \right] {}_0d_q\eta \right) + (1 - t_1) \left(\int_0^{t_2} 1^{\mu} \, {}_0d_q\eta \right)^{\frac{1}{\mu}}$$
$$\left. \times \left(\int_0^{t_2} \left[h\left(\eta^{\delta}\right) \left| {}_rD_qP(\xi) \right|^{\beta} + mh\left(1 - \eta^{\delta}\right) \left| {}_rD_qP\left(\frac{r}{m}\right) \right|^{\beta} \right] {}_0d_q\eta \right) \right\}, \tag{12}$$

for all $t_1, t_2 \in [0, 1]$ with $\beta^{-1} + \mu^{-1} = 1$.

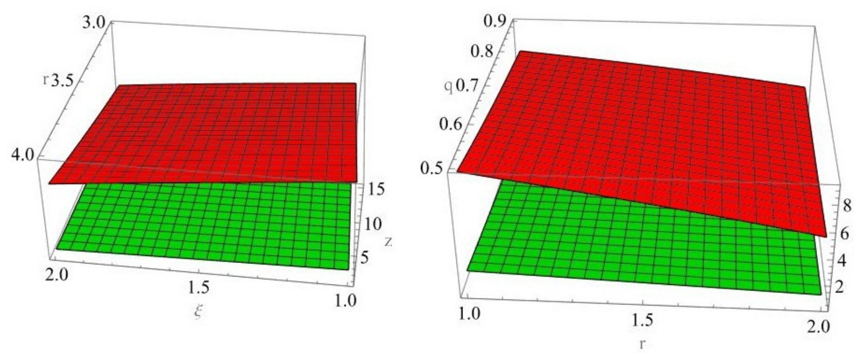
Proof. By making use of Lemma 2.1 and Hölder's inequality, we have

$$|\Omega| \leq (\xi - r)\Bigg\{\left(\int_0^1 |q\eta - (1-t_1t_2)|^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\left(\int_0^1 \left|{}_rD_qP\big(\eta\xi + (1-\eta)r\big)\right|^{\beta}\ {}_0d_q\eta\right)^{\frac{1}{\beta}}$$
$$+(1-t_1)\left(\int_0^{t_2} 1^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\left(\int_0^{t_2} \left|{}_rD_qP\big(\eta\xi + (1-\eta)r\big)\right|^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\beta}}\Bigg\}.$$

Since $\left|{}_rD_qP\right|^{\beta}$, $\beta > 1$ is $(\delta, m, h)-$convex, so we have

$$|\Omega| \leq (\xi - r)\Bigg\{\left[\int_0^1 |q\eta - (1-t_1t_2)|^{\mu}\ {}_0d_q\eta\right]^{\frac{1}{\mu}}$$
$$\times\left(\int_0^1 \left[h(\eta^{\delta})\left|{}_rD_qP(\xi)\right|^{\beta} + mh(1-\eta^{\delta})\ \left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]\ {}_0d_q\eta\right.$$
$$+(1-t_1)\left(\int_0^{t_2} 1^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\left(\int_0^{t_2}\left[h(\eta^{\delta})\left|{}_rD_qP(\xi)\right|^{\beta} + mh(1-\eta^{\delta})\ \left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]\ {}_0d_q\eta\right)\Bigg\}.$$

Example 3.3. *We calculate all the integrals involved in* (12) *for the function discussed in above examples, we draw few graphs for various values of parameters involved and present in Figure* 3*. Left hand side of* (12) *is represented by green color and right side by red color. Clearly,* (12) *is justified by graphs.*



(A) Here $t_1 = 0.1, t_2 = 0.5, q = 0.7, \xi \in [1, 2]$ and $r \in [3, 4]$ (B) Here $t_1 = 0.1, t_2 = 0.5, \xi = 3, r \in [1, 2]$ and $q \in [0.5, 0.9]$

Figure 3: Graphical illustration of (12).

Corollary 3.3. *Setting $h(t) = t^s$ in Theorem 3.3, we attain the bounds for $(\delta, s, m)-$convex functions as:*

$$|\Omega| \leq (\xi - r)\left\{\left(\int_0^1 |q\eta - (1-t_1t_2)|^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\right.$$
$$\times\left(\int_0^1 \left[\eta^{s\delta}\left|{}_rD_qP(\xi)\right|^{\beta} + m\left(1-\eta^{\delta}\right)^{s}\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]\ {}_0d_q\eta\right) + (1-t_1)\left(\int_0^{t_2} 1^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}$$
$$\left.\times\left(\int_0^{t_2} \left[\eta^{s\delta}\left|{}_rD_qP(\xi)\right|^{\beta} + m\left(1-\eta^{\delta}\right)^{s}\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]\ {}_0d_q\eta\right)\right\}. \tag{13}$$

Remark 3.3.

1. *Substituting $\delta = 1$ in above corollary, we attain the bounds for $(s, m)-$convex functions.*
2. *By choosing $\delta = 1$, the result coincide with Theorem 6 proved in [1].*

□

Example 3.4. *Let $P(x) = \dfrac{x^2}{4}$ and we choose $h(\eta) < \eta$, then clearly $P(x)$ satisfies all the conditions of Theorem 3.1. Now, choosing $r = 1$, $\xi = 2$, $t_1 = \dfrac{1}{2} = t_2$. We have $\left|{}_rD_qP\left(\dfrac{r}{m}\right)\right| = \dfrac{1+q+m-mq}{4m}$ and $|{}_rD_qP(\xi)| = \dfrac{3+q}{4}$. Now, using the definition of quantum integrals, we evaluate all the integrals involved in R.H.S of (3.1) as follows;*

1. $\displaystyle\int_0^{\frac{1}{2}} \left|qt - \frac{1}{4}\right| \frac{t^{\delta}}{2}\ {}_0d_qt$

$$= \begin{cases} \dfrac{(1-q)(1-2q+q^{\delta+2})}{2^{\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad 0 < q < \dfrac{1}{2}, \\[2ex] \dfrac{(1-q)\left[(1-2^{\delta}) + q(2^{\delta+1}-1) - 2^{\delta}q^{\delta+2}\right]}{2^{2\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad \dfrac{1}{2} \leq q \leq 1. \end{cases}$$

2. $\displaystyle\int_0^{1} \frac{t^{\delta}}{2}\left|qt - \frac{3}{4}\right|\ {}_0d_qt$

$$= \begin{cases} \dfrac{(1-q)(3-4q+q^{\delta+2})}{8(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad 0 < q < \dfrac{3}{4}, \\[2ex] \dfrac{(1-q)\left[(3^{\delta+2} - 3\ldots 2^{2\delta+1}) + q(2^{2\delta+3} - 3^{\delta+2}) - 2^{2\delta+1}q^{\delta+2}\right]}{2^{2\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad \dfrac{3}{4} < q \leq 1. \end{cases}$$

3. $\displaystyle\int_0^{\frac{1}{2}} \frac{t^{\delta}}{2}\left|qt - \frac{3}{4}\right|\ {}_0d_qt = \frac{(1-q)(q^{\delta+2}+2q-3)}{2^{\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}.$

4. $$\int_0^{\frac{1}{2}} \left| qt - \frac{1}{4} \right| \frac{1-t^\delta}{2} \, {}_0d_qt$$

$$= \begin{cases} \frac{1-q}{16}\left[-\frac{1}{1+q} - \frac{2q-1-q^{\delta+2}}{2^\delta(1-q^{\delta+1})(1-q^{\delta+2})}\right], & \textit{if} \quad 0 < q < \frac{1}{2}, \\ \frac{q}{16(1+q)} + \frac{(1-q)\left[(2^\delta-1)+q(1-2^{\delta+1})+2^\delta q^{\delta+2}\right]}{2^{2\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad \frac{1}{2} \le q \le 1. \end{cases}$$

5. $$\int_0^1 \frac{1-t^\delta}{2} \left| qt - \frac{3}{4} \right| \, {}_0d_qt$$

$$= \begin{cases} \frac{3-q}{8(1+q)} + \frac{(1-q)(4q-3-q^{\delta+2})}{8(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad 0 < q < \frac{3}{4}, \\ \frac{3+2q}{16(1+q)} - \frac{(1-q)\left[(3^{\delta+2}-3\ldots 2^{2\delta+1})+q(2^{2\delta+3}-3^{\delta+2})-2^{2\delta+1}q^{\delta+2}\right]}{2^{2\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}, & \textit{if} \quad \frac{3}{4} \le q \le 1. \end{cases}$$

6. $$\int_0^{\frac{1}{2}} \frac{1-t^\delta}{2} \left| qt - \frac{3}{4} \right| \, {}_0d_qt = \frac{3}{16(1+q)} + \frac{5q-3-2q^2+q^{\delta+2}-q^{\delta+3}}{2^{\delta+4}(1-q^{\delta+1})(1-q^{\delta+2})}.$$

Now, we are in position to construct a comparison table of R.H.S of (3.1) *and R.H.S of Theorem* 4 *in* [1] *in the following manner;*

Table 1: Comparison of estimates.

Sr. No	q	m	δ	(h, m)	(δ, h, m)
1	0.4	0.9	0.1	0.3816	0.1908
2	0.4	0.9	0.2	0.3504	0.1749
3	0.4	0.9	0.3	0.3238	0.1619
4	0.4	0.9	0.4	0.3020	0.1510
5	0.4	0.9	0.5	0.2838	0.1419
6	0.4	0.95	0.5	0.2814	0.1407
7	0.4	0.955	0.5	0.2812	0.1406
8	0.4	0.9555	0.5	0.2810	0.1405
9	0.4	0.96	0.5	0.2808	0.1404
10	0.4	0.97	0.5	0.2804	0.1402

Remark 3.4. *From Table 1, it is evident that by using the condition $h(\eta) < \eta$ along with Theorem 3.1, we offer better estimates than in Theorem* 4 *presented in* [1]. *Moreover, as m and δ approach to* 1 *from left, we achieve better and refined estimates. Similarly, the results presented in this investigation indicate the refinement of bounds under the condition mentioned in example. Now, for more better understanding the refinement of estimates, we make use of graphical representation of estimates achieved through $(h, m)-$convexity and $(\delta, h, m)-$convexity. We choose $q = 0.4$, $\delta = 0.8$ and $m \in [0.1, 0.5]$. Green color shows estimates of*

(δ, h, m) and red color is used for $(h, m)-$convexity. Next, graphical representation shows the refinement of bounds in Figure 4.

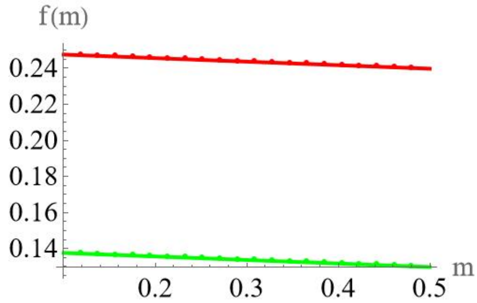


Figure 4: The refinement of estimates.

4 Applications to Midpoint-like Results

In this section, we re-frame different midpoint-like results in order to focus on the usability of the results as application point of view as well as for better understanding of these results.

4.1 $q-$midpoint-like results

(i) Consider Theorem 3.1, using $t_1 = 0$ and $t_2 = \dfrac{1}{1+q}$, we obtain

$$\begin{aligned}
&\left| P\left(\frac{qr+\xi}{1+q}\right) - \frac{1}{\xi - r}\int\limits_r^\xi P(x)\,{}_r d_q x \right| \\
&\leq (\xi - r)\left\{\left[\int\limits_0^{\frac{1}{1+q}} h\big(\eta^\delta\big)|q\eta|\,{}_0 d_q\eta + \int\limits_0^1 h\big(\eta^\delta\big)|q\eta - 1|\,{}_0 d_q\eta - \int\limits_0^{\frac{1}{1+q}} h\big(\eta^\delta\big)|q\eta - 1|\,{}_0 d_q\eta\right]\right. \\
&\quad \times \big|\,{}_r D_q P(\xi)\big| + m\left[\int\limits_0^{\frac{1}{1+q}} h\big(1-\eta^\delta\big)|q\eta|\,{}_0 d_q\eta + \int\limits_0^1 h\big(1-\eta^\delta\big)|q\eta - 1|\,{}_0 d_q\eta\right. \\
&\quad \left.\left. - \int\limits_0^{\frac{1}{1+q}} h\big(\eta^\delta\big)|q\eta - 1|\,{}_0 d_q\eta\right]\left|\,{}_r D_q P\left(\frac{r}{m}\right)\right|\right\}, \qquad (14)
\end{aligned}$$

which is the $q-$midpoint-like inequality for $(\delta, m, h)-$convex functions.

(ii) Taking $h(t) = t^s$ in (14), we have q-midpoint-like result for $(\delta, s, m)-$convex functions as,

$$\begin{aligned}
&\left| p\left(\frac{qr+\xi}{1+q}\right) - \frac{1}{\xi - r}\int\limits_r^\xi P(x)\ {}_r d_q x \right| \\
&\quad \leq (\xi - r)\left\{ \left[\int\limits_0^{\frac{1}{1+q}} \left(\eta^\delta\right)^s |q\eta|\ {}_0 d_q\eta + \int\limits_0^1 \left(\eta^\delta\right)^s |q\eta - 1|\ {}_0 d_q\eta - \int\limits_0^{\frac{1}{1+q}} \left(\eta^\delta\right)^s |q\eta - 1|\ {}_0 d_q\eta \right] \right. \\
&\qquad \times \left| {}_r D_q p(\xi)\right| + m\left[\int\limits_0^{\frac{1}{1+q}} \left(1-\eta^\delta\right)^s |q\eta|\ {}_0 d_q\eta + \int\limits_0^1 \left(1-\eta^\delta\right)^s |q\eta - 1|\ {}_0 d_q\eta \right. \\
&\qquad \left.\left. - \int\limits_0^{\frac{1}{1+q}} \left(\eta^\delta\right)^s |q\eta - 1|\ {}_0 d_q\eta \right] \left| {}_r D_q p\left(\frac{r}{m}\right)\right| \right\}.
\end{aligned} \tag{15}$$

(iii) For $\delta = 1$ in (15), we have $q-$midpoint-like result for $(s, m)-$convex functions.

(iv) Let $t_1 = \dfrac{1}{2} = t_2$ in Theorem 3.1, we have

$$\begin{aligned}
&\left| \frac{1}{2}\left[\frac{p(\xi)+p(r)}{2} + P\left(\frac{\xi + r}{2}\right)\right] - \frac{1}{\xi - r}\int\limits_r^\xi P(x)\ {}_r d_q x \right| \\
&\quad \leq (\xi - r)\left\{ \frac{1}{2}\left[\int\limits_0^{\frac{1}{2}} h(\eta^\delta)\ {}_0 d_q\eta + \int\limits_0^1 h(\eta^\delta)\left| q\eta - \frac{3}{4}\right|\ {}_0 d_q \right] \left| {}_r D_q p(\xi)\right| \right. \\
&\qquad \left. + m\left[\int\limits_0^{\frac{1}{2}} h(1-\eta^\delta)\ {}_0 d_q\eta + \int\limits_0^1 h(1-\eta^\delta)\left| q\eta - \frac{3}{4}\right|\ {}_0 d_q\eta \right] \left| {}_r D_q p\left(\frac{r}{m}\right)\right| \right\}.
\end{aligned} \tag{16}$$

(v) For $m = 1$ in (14), we recapture Corollary 2.3(i).

(vi) Taking $m = 1$ in above equation, we obtain Theorem 3 in [3].

(vii) Consider Theorem 3.2, substituting $t_1 = 0$ and $t_2 = \dfrac{1}{1+q}$, we have

$$\begin{aligned}
&\left| p\left(\frac{qr+\xi}{1+q}\right) - \frac{1}{\xi - r}\int\limits_r^\xi P(x)\ {}_r d_q x \right| \\
&\quad \leq (\xi - r)\left\{ \left[\int\limits_0^1 |q\eta - 1)|\ {}_0 d_q\eta \right]^{1-\frac{1}{\beta}} \times \left[\left(\int\limits_0^1 h(\eta^\delta)|q\eta - 1 - |\ {}_0 d_q\eta \right) \left| {}_r D_q P(\xi)\right|^\beta \right.\right.
\end{aligned}$$

$$+m\left(\int_0^1 h(1-\eta^\delta)|q\eta-1|_0 d_q\eta\right)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\Bigg]^{\frac{1}{\beta}}+\left(1-\left(\frac{1}{1+q}\right)^{1-\frac{1}{\beta}}\right)$$
$$\times\left[\left(\int_0^{\frac{1}{1+q}} h(\eta^\delta)\ {}_0d_q\eta\right)\left|{}_rD_qP(\xi)\right|^\beta+m\left(\int_0^{\frac{1}{1+q}} h(1-\eta^\delta)\ {}_0d_q\eta\right)\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]^{\frac{1}{\beta}}\Bigg\},$$

which is the $q-$midpoint-like result for $\beta \geq 1$.

(viii) Taking $h(t) = t^s$ in above inequality, we obtain $q-$midpoint-like results for $(\delta, s, m)-$convex function for $\beta \geq 1$, and by choosing $\delta = 1$, we retrieve the bounds for the class of $(s, m)-$convex functions.

(ix) Taking $\delta = 1$ in above inequality, we obtain Corollary 2(i) in [1].

(x) Consider Theorem 3.3, substituting $t_1 = 0$ and $t_2 = \dfrac{1}{1+q}$, we have

$$\left|p\left(\frac{qr+\xi}{1+q}\right)-\frac{1}{\xi-r}\int_r^\xi P(x)\ {}_rd_qx\right|$$
$$\leq(\xi-r)\left\{\left(\int_0^1|q\eta-1)|^\mu\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\times\right.$$
$$\left(\int_0^1\left[h(\eta^\delta)\left|{}_rD_qP(\xi)\right|^\beta+mh(1-\eta^\delta)\ \left|{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]\ {}_0d_q\eta\right)$$
$$\left.+\left(\int_0^{\frac{1}{1+q}}1^\mu\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{1+q}}\left[h(\eta^\delta)\left|{}_rD_qP(\xi)\right|^\beta+mh(1-\eta^\delta)\ \left|{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]\ {}_0d_q\eta\right)\right\},\tag{17}$$

which is $q-$midpoint-like result for $\beta > 1$.

(xi) In particular, putting $h(t) = t^s$ in (17), we have

$$\left|p\left(\frac{qr+\xi}{1+q}\right)-\frac{1}{\xi-r}\int_r^\xi P(x)\ {}_rd_qx\right|$$
$$\leq(\xi-r)\left\{\left(\int_0^1|q\eta-1)|^\mu\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\right.$$
$$\left(\int_0^1\left[\eta^{s\delta}\left|{}_rD_qP(\xi)\right|^\beta+m(1-\eta^\delta)^s\ \left|{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right]\ {}_0d_q\eta\right)$$

$$+\left(\int_0^{\frac{1}{1+q}} 1^{\mu}\ {}_0d_q\eta\right)^{\frac{1}{\mu}}\left(\int_0^{\frac{1}{1+q}}\left[\eta^{s\delta}\left|{}_rD_qP(\xi)\right|^{\beta}+m\left(1-\eta^{\delta}\right)^{s}\left|{}_rD_qP\left(\frac{r}{m}\right)\right|^{\beta}\right]{}_0d_q\eta\right)\Bigg\}, \tag{18}$$

which are the bounds for $(\delta, s, m)-$convex function with $\beta > 1$.

(xii) For $\delta = 1$ in (18), we obtain bounds for the class of $(s, m)-$convex functions with $\beta > 1$.

(xiii) For $\delta = 1$, we obtain Corollary 2(i) for Theorem 6 in [1].

4.2 $q-$Simpson-like results

(i) Taking $t_1 = \dfrac{1}{3}$ and $t_2 = \dfrac{1}{1+q}$ in Theorem 3.1, we have the inequality,

$$\left|\frac{1}{3}\left[\frac{qp(r)+p(\xi)}{1+q}+2p\left(\frac{qr+\xi}{1+q}\right)\right]-\frac{1}{\xi-r}\int_r^{\xi}P(x)\ {}_rd_qx\right|$$
$$\leq(\xi-r)\left\{\left[\int_0^{\frac{1}{1+q}}h\left(\eta^{\delta}\right)\left|q\eta-\frac{q}{3(1+q)}\right|{}_0d_q\eta+\int_0^{1}h\left(\eta^{\delta}\right)\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right.\right.$$
$$\left.-\int_0^{\frac{1}{1+q}}h\left(\eta^{\delta}\right)\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right]\left|{}_rD_qp(\xi)\right|$$
$$+m(\xi-r)\left[\int_0^{\frac{1}{1+q}}h\left(1-\eta^{\delta}\right)\left|q\eta-\frac{q}{3(q+1)}\right|{}_0d_q\eta+\int_0^{1}h\left(1-\eta^{\delta}\right)\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right.$$
$$\left.\left.-\int_0^{\frac{1}{1+q}}h\left(\eta^{\delta}\right)\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right]\left|{}_rD_qp\left(\frac{r}{m}\right)\right|\right\}, \tag{19}$$

which is $q-$Simpson-like inequality for $(\delta, m, h)-$convex functions.

(ii) Taking $h(t) = t^s$, we achieve bounds for $q-$Simpson-like inequality in the setting of $(\delta, s, m)-$convex function in the following manner,

$$\left|\frac{1}{3}\left[\frac{qp(r)+p(\xi)}{1+q}+2p\left(\frac{qr+\xi}{1+q}\right)\right]-\frac{1}{\xi-r}\int_r^{\xi}P(x)\ {}_rd_qx\right|$$
$$\leq(\xi-r)\left\{\left[\int_0^{\frac{1}{1+q}}\eta^{s\delta}\left|q\eta-\frac{q}{3(1+q)}\right|{}_0d_q\eta+\int_0^{1}\eta^{s\delta}\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right.\right.$$
$$\left.-\int_0^{\frac{1}{1+q}}\left(\eta^{\delta}\right)^{s}\left|q\eta-\frac{3q+2}{3(q+1)}\right|{}_0d_q\eta\right]\left|{}_rD_qp(\xi)\right|$$

$$+ m(\xi - r)\left[\int_0^{\frac{1}{1+q}} (1-\eta^\delta)^s \left|q\eta - \frac{q}{3(q+1)}\right| \,_0d_q\eta + \int_0^1 (1-\eta^\delta)^s \left|q\eta - \frac{3q+2}{3(q+1)}\right| \,_0d_q\eta \right.$$
$$\left. - \int_0^{\frac{1}{1+q}} (\eta^\delta)^s \left|q\eta - \frac{3q+2}{3(q+1)}\right| \,_0d_q\eta\right] \left| \,_rD_q p\left(\frac{r}{m}\right)\right| \Bigg\}. \tag{20}$$

(iii) for $\delta = 1$ in (20), we regain the estimates for $(s, m)-$convex functions.

(iv) For $\delta = 1 = m$ in (19), we obtain Corollary 1(iii) of [1].

(v) For $\delta = 1 = m$ and $q \to 1^-$in (19), we obtain Corollary 1 of [2].

(vi) Taking $t_1 = \dfrac{1}{3}$ and $t_2 = \dfrac{1}{1+q}$ in Theorem 3.2, we have the inequality,

$$\left|\frac{1}{3}\left[\frac{qp(r)+p(\xi)}{1+q} + 2p\left(\frac{qr+\xi}{1+q}\right)\right] - \frac{1}{\xi - r}\int_r^\xi P(x) \,_rd_qx\right|$$
$$\leq (\xi - r)\left\{\left[\int_0^1 \left|q\eta - \frac{3q+2}{3(q+1)}\right| \,_0d_q\eta\right]^{1-\frac{1}{\beta}} \times \left[\left| \,_rD_qP(\xi)\right|^\beta \int_0^1 h(\eta^\delta)\left|q\eta - \frac{3q+2}{3(q+1)}\right| \,_0d_q\eta\right.\right.$$
$$\left. + m\left| \,_rD_qP\left(\frac{r}{m}\right)\right|^\beta \int_0^1 h(1-\eta^\delta)\left|q\eta - \frac{3q+2}{3(q+1)}\right| \,_0d_q\eta\right]^{\frac{1}{\beta}} + \frac{2}{3}\left(\frac{1}{q+1}\right)^{1-\frac{1}{\beta}}$$
$$\left.\times\left[\left| \,_rD_qP(\xi)\right|^\beta \int_0^{t_2} h(\eta^\delta) \,_0d_q\eta + m\left| \,_rD_qP\left(\frac{r}{m}\right)\right|^\beta \int_0^{t_2} h(1-\eta^\delta) \,_0d_q\eta\right]^{\frac{1}{\beta}}\right\}, \tag{21}$$

which is the $q-$Simpson-like integral inequality in the sense of $(\delta, m, h)-$convex functions.

(vii) Setting $\delta = 1$ in (21), we obtain Corollary 2(ii) for Theorem 5 in [1].

(viii) Putting $t_1 = \dfrac{1}{3}$ and $t_2 = \dfrac{1}{1+q}$ in Theorem 3.3, we have the following inequality,

$$\left|\frac{1}{3}\left[\frac{qp(r)+p(\xi)}{1+q} + 2p\left(\frac{qr+\xi}{1+q}\right)\right] - \frac{1}{\xi - r}\int_r^\xi P(x) \,_rd_qx\right|$$
$$\leq (\xi - r)\left\{\left(\int_0^1 \left|q\eta - \frac{3q+2}{3(q+1)})\right|^\mu \,_0d_q\eta\right)^{\frac{1}{\mu}}\right.$$
$$\left(\int_0^1 \left[h(\eta^\delta)\left| \,_rD_qP(\xi)\right|^\beta + mh(1-\eta^\delta)\ \left| \,_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right] \,_0d_q\eta\right) + \frac{2}{3}\left(\int_0^{\frac{1}{1+q}} 1^\mu \,_0d_q\eta\right)^{\frac{1}{\mu}}$$
$$\left.\times\left(\int_0^{\frac{1}{1+q}} \left[h(\eta^\delta)\left| \,_rD_qP(\xi)\right|^\beta + mh(1-\eta^\delta)\ \left| \,_rD_qP\left(\frac{r}{m}\right)\right|^\beta\right] \,_0d_q\eta\right)\right\}. \tag{22}$$

(ix) Setting $\delta = 1$ in (22), we obtain Corollary 2(ii) for Theorem 6 in [1].

4.3 Averaged midpoint-trapezoid-like and $q-$trapezoid like results

(i) Let $t_1 = \dfrac{1}{2}$ and $t_2 = \dfrac{1}{1+q}$ in Theorem 3.1, we are left with the inequality,

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{qp(r)+p(\xi)}{1+q}+p\left(\frac{qr+\xi}{1+q}\right)\right]-\frac{1}{\xi-r}\int_r^\xi P(x)\,{}_rd_qx\right| \\
&\leq (\xi-r)\left[\int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta-\frac{q}{2(1+q)}\right|\,{}_0d_q\eta+\int_0^1 h(\eta^\delta)\left|q\eta-\frac{2q+1}{2(q+1)}\right|\,{}_0d_q\eta\right. \\
&\left.-\int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta-\frac{2q+1}{2(q+1)}\right|\,{}_0d_q\eta\right]\left|\,{}_rD_qp(\xi)\right|+m(\xi-r) \\
&\times\left[\int_0^{\frac{1}{1+q}} h(1-\eta^\delta)\left|q\eta-\frac{q}{2(q+1)}\right|\,{}_0d_q\eta+\int_0^1 h(1-\eta^\delta)\left|q\eta-\frac{2q+1}{2(1+q)}\right|\,{}_0d_q\eta\right. \\
&\left.-\int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta-\frac{2q+1}{2(1+q)})\right|\,{}_0d_q\eta\right]\left|\,{}_rD_qp\left(\frac{r}{m}\right)\right|, \qquad (23)
\end{aligned}$$

which is the averaged midpoint-trapezoid-like inequality for $(\delta, m, h)-$convex functions.

(ii) For $\delta = 1$ in (23), we obtain Corollary 1(iv) of [1].

(iii) In (23), by putting $m = 1 = \delta$ and $q \to 1^-$, we obtain Corollary 3.4 in [43].

(iv) Let $t_1 = \dfrac{1}{2}$ and $t_2 = \dfrac{1}{1+q}$ in Theorem 3.2, we are left with the inequality,

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{qp(r)+p(\xi)}{1+q}+p\left(\frac{qr+\xi}{1+q}\right)\right]-\frac{1}{\xi-r}\int_r^\xi P(x)\,{}_rd_qx\right| \\
&\leq (\xi-r)\left\{\left(\int_0^1\left|q\eta-\frac{2q+1}{2(q+1)}\right|\,{}_0d_q\eta\right)^{1-\frac{1}{\beta}}\left[\left|\,{}_rD_qP(\xi)\right|^\beta\int_0^1 h(\eta^\delta)\left|q\eta-\frac{2q+1}{2(q+1)}\right|\,{}_0d_q\eta\right.\right. \\
&\left.+m\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\int_0^1 h(1-\eta^\delta)\left|q\eta-\frac{2q+1}{2(q+1)}\right|_0 d_q\eta\right]^{\frac{1}{\beta}}+\frac{1}{2}\left(\frac{1}{1+q}\right)^{1-\frac{1}{\beta}} \\
&\left.\times\left[\left|\,{}_rD_qP(\xi)\right|^\beta\int_0^{\frac{1}{1+q}} h(\eta^\delta)\,{}_0d_q\eta+m\left|\,{}_rD_qP\left(\frac{r}{m}\right)\right|^\beta\int_0^{\frac{1}{1+q}} h(1-\eta^\delta)\,{}_0d_q\eta\right]^{\frac{1}{\beta}}\right\},
\end{aligned}$$

which is the averaged midpoint-trapezoid-like integral inequality for $(\delta, h, m)-$convexity for $\beta \geq 1$.

Remark 4.1. *We obtain averaged midpoint-trapezoid-like integral inequality for* $(\delta, h, m)-$*convexity for* $\beta > 1$ *by setting* $t_1 = \frac{1}{2}$ *and* $t_2 = \frac{1}{1+q}$ *in Theorem 3.2.*

(v) Substituting $t_1 = 1$ and $t_2 = \frac{1}{1+q}$ in Theorem 3.1, we have

$$\begin{aligned}
&\left|\frac{qp(r)+p(\xi)}{1+q} - \frac{1}{\xi - r}\int_r^\xi P(x)\ {}_r d_q x\right| \\
&\leq (\xi - r)\left\{\left[\int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta - \frac{q}{1+q}\right|\ {}_0 d_q\eta + \int_0^1 h(\eta^\delta)\left|q\eta - \frac{q}{q+1}\right|\ {}_0 d_q\eta \right.\right.\\
&\quad \left. - \int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta - \frac{q}{q+1}\right|\ {}_0 d_q\eta\right] \left|\ {}_r D_q p(\xi)\right| \\
&\quad + m(\xi - r)\left[\int_0^{\frac{1}{1+q}} h(1-\eta^\delta)\left|q\eta - \frac{q}{q+1}\right|\ {}_0 d_q\eta + \int_0^1 h(1-\eta^\delta)\left|q\eta - \frac{q}{1+q}\right|\ {}_0 d_q\eta \right.\\
&\quad \left.\left. - \int_0^{\frac{1}{1+q}} h(\eta^\delta)\left|q\eta - \frac{q}{1+q}\right|\ {}_0 d_q\eta\right] \left|\ {}_r D_q p\left(\frac{r}{m}\right)\right|\right\},
\end{aligned} \tag{24}$$

which is $q-$trapezoid like inequality for $(\delta, m, h)-$convex functions.

(vi) For $\delta = 1$ in (24), we obtain Corollary 1(v) of [1].

(vii) In (24), utilizing $\delta = 1 = m$, we obtain Theorem 4.1 in [46].

(viii) Choosing $t_1 = 1$ and $t_2 = \frac{1}{1+q}$ in Theorem 3.2, we have

$$\begin{aligned}
&\left|\frac{qp(r)+p(\xi)}{1+q} - \frac{1}{\xi - r}\int_r^\xi P(x)\ {}_r d_q x\right| \\
&\leq (\xi - r)\left\{\left(\int_0^1 \left|q\eta - \frac{q}{q+1}\right|\ {}_0 d_q\eta\right)^{1-\frac{1}{\beta}} \left[\left|\ {}_r D_q P(\xi)\right|^\beta \int_0^1 h(\eta^\delta)\left|q\eta - \frac{q}{q+1}\right|\ {}_0 d_q\eta \right.\right.\\
&\quad \left.\left. + m\left|\ {}_r D_q P\left(\frac{r}{m}\right)\right|^\beta \int_0^1 h(1-\eta^\delta)\left|q\eta - \frac{q}{q+1}\right|_0 d_q\eta\right]^{\frac{1}{\beta}}\right\},
\end{aligned} \tag{25}$$

which is the $q-$trapezoid-like inequality for $\beta \geq 1$.

(ix) For $\delta = 1$ in (25), we obtain Corollary 2(iv) of Theorem 5 in [1].

(x) Setting $t_1 = 1$ and $t_2 = \frac{1}{1+q}$ in Theorem 3.3, we have

$$\left| \frac{qp(r) + p(\xi)}{1+q} - \frac{1}{\xi - r} \int_r^\xi P(x) \; _rd_qx \right|$$
$$\leq (\xi - r) \left\{ \left(\int_0^1 \left| q\eta - \frac{q}{q+1} \right|^\mu \; _0d_q\eta \right)^{\frac{1}{\mu}} \right.$$
$$\left. \times \left(\int_0^1 \left[h(\eta^\delta) \left| \; _rD_qP(\xi) \right|^\beta + mh(1-\eta^\delta) \; \left| \; _rD_qP\left(\frac{r}{m}\right) \right|^\beta \right] \; _0d_q\eta \right) \right\}. \tag{26}$$

(xi) For $\delta = 1$ in (26), we obtain Corollary 2(iv) of Theorem 6 in [1].

Remark 4.2. *As a special case, if we utilize $h(t) = t^s$ in averaged midpoint-trapezoid-like and $q-$trapezoid-like results, one can establish estimates for the class of $(\delta, s, m)-$convex functions and $(s, m)-$convex functions for $\delta = 1$.*

5 Conclusion

Since its creation in the late 19th century, the Hermite-Hadamard inequality has been a fundamental component of convex function analysis. This inequality, which was inspired by the writings of Jacques Hadamard and Charles Hermite, offered a sophisticated way to bound the average value of a convex function between its values at the interval's midpoint and endpoints. This study reveals the investigation of classical (1) by using generalized convex function via quantum integrals. The resulting constraints were extended with the help of Hölder's inequality and the power mean inequality, demonstrating how classical inequalities remain essential tools even in complex generalized scenarios. As special cases of the presented results, the derivation of $q-$midpoint-like, Simpson-like, averaged midpoint-like, and trapezoid-type integral inequalities are some of the study's main findings. These findings combine and generalize a number of established inequalities inside a single framework.

For different choices of the parameters, we included the graphical representation of the inequalities (5), (7), and (12) that verified our presented results. In order to show the coherence with already existing literature, we compare our result with previous estimates by constructing Table 1. Clearly, Table 1 shows the estimates presented in this analysis are refinement. To further elaborate, we extend this comparison with help of graphical representation through Figure 4. Finally, to highlight the relevance of the results, two fresh applications are offered for support. We anticipate that the new inequalities and applications discussed here will inspire more study and advancement in the area, leading to a better comprehension of quantum integrals and how they relate to integral inequalities and convexity. This article emphasizes how fundamental mathematical concepts like convexity and the Hermite-Hadamard inequality remain relevant even when they are modified to meet contemporary contexts and problems.

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